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Bigraphs Reloaded

a presheaf presentation

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Technical report
UDMI/01/2013/RR

2013-03-10

Bigraphs reloaded: a presheaf presentation

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Abstract

Milner's supported (pre)categories have been introduced as the formal framework for the development of the theory of bigraphs and bigraphical reactive systems. We give a new presentation of supported precategories as particular presheaves inspired to profunctors. These presheaf categories are naturally endowed with two monoidal structures, representing composition and tensor product in supported (pre)categories.

This new categorical setting allows for new results about compositional graphical structures (not only bigraphs), taking advantage of the well-established theory of presheaf categories. As an application, we show that bigraphs and similar graphical objects can be characterized as free algebras arising from the monoidal structures of the presheaf category. Moreover, we show how sorting disciplines can be expressed formally using the internal language of these topoi.

1 Introduction

Robin Milner introduced *supported (pre)categories* as a generalization of the notion of category, by relaxing the requirement that each pair of compatible arrows can be composed. Each morphism is associated with a finite set (its *support*); two morphisms compose if and only if their supports are disjoint. Supported precategories have been used especially in the theory of reactive systems [19], leading to important results. In particular, *bigraphs* (and similar compositional graphical structures) can be formalized as supported precategories [18], akin to the presentation of term algebras as Lawvere theories: objects are interfaces, morphisms are bigraphs and supports are internal names [25, Def. 2.17]; composition captures bigraph composition along a shared interface. This formalization has been the fertile soil for the wide range of results offered by the bigraphical framework [18, 22, 25], such as the so called *IPO construction* for the synthesis of minimal labeled transition systems for bigraphical reactive systems.

Nevertheless, this presentation is not well suited for studying several issues and constructions that we are often interested in, when dealing with process calculi and dynamic systems. Examples are the *initial semantics* and *bialgebraic semantics*, as in [31], where the syntax of systems has to be characterized as the initial algebra of a endofunctor induced by the signature. So the whole class of bigraphs has to be represented as a single object of a suitable category, instead of being a category on its own.

As Milner pointed out in [25, Cap. 6], another crucial notion is that of *sorting discipline.* Sortings should be seen as an integral part of the formalization of any language, since non-well formed agents must be ruled out in order to obtain a faithful formalization. Sortings are defined as predicates over supported morphisms, but we need a full fledged language, a logical setting, and a formal interpretation for these predicates.

Another issue concerns the *refinement* of signatures, i.e. what happens when we change the signature, e.g. by adding/removing/merging controls? How are the syntax and semantics affected? To investigate these issues we need a formal setting for representing *operations* on signatures, and studying how these operations are reflected to the corresponding theories of bigraphs.

In fact, one quickly realizes that bigraphs are just one example in a countless family of "pluri-graphical" structures. Similar but different structures are directed bigraphs [16] and bigraphs with sharing [5]. Trying to find the "most general" pluri-graphical structure is hopeless. More reasonably, we can aim to define a scenario where generic graphical structures can be combined and merged, in the way hypergraphs and trees combine to yield bigraphs.

In this paper, we propose such a general scenario for addressing these issues. Our main aim is to define a category where each object represents a whole class of supported terms or graphical structures (such as link/place graphs and bigraphs) generated by a given signature. Morphisms between these objects represent transformations between different graphical languages. Objects can be defined as initial algebras of suitable endofunctors, thus yielding naturally algebraic description for these graphical structures. This setting will allow to show how different graphical structures can be combined together (also beyond bigraphs), and will also accomodate the notions of sortings as predicates and subobject.

The main technical step towards this aim is to consider *presheaf categories*, i.e. categories of Set-valued functors over an index category of *types*. As for normal term theories, these types have to convey the informations exposed by graphical structures needed for defining composition and juxtaposition. As an example, a bigraph $G = (V, E, ctrl, link, prnt) : \langle n, X\rangle \to \langle m, Y\rangle$ exposes three distinct informations: the inner and the outer interfaces $\langle n, X\rangle$, $\langle m, Y\rangle$, and the *support*, i.e. the set of its nodes. The index category has to cover these informations, and the operations on them. The key point is to see presheaves as "indexed sets" of morphisms (e.g., bigraphs), whose domain, codomain and support are given by the index; then, we can take advantage of many standard constructions in presheaf categories for building new data structures from existing ones.

This setting turns out to be very flexible, as we can capture several notions of composition for arrows by changing the structure of the index category. This enables a great degree of freedom without the loss of important tools and results offered by the theory of presheaves and profunctors. Moreover, as any presheaf category is also a topos, we can take advantage of its internal language for defining predicates, and subobjects.

The rest of the paper is organized as follows. In Section 2 we present the formal general settings of "profunctors with support" as particular presheaves; we show how these correspond to supported (pre)categories, and some useful results. In Section 3 we put this theory at work, by giving a new presentation of bigraphs, starting with their components (*i.e.* signatures, link and place graphs), how these combine yielding bigraphical structures, and giving an algebraic presentation—all this, within the *same* environment category.

Then, in Section 4, we take into account the notion of sorting. We will see that we can represent sortings as suitable predicates (called *monoidal sortings*) using the internal language of the presheaf category. As an example, we show that sorted bigraphs form a subpresheaf (a subobject) closed under decompositions.

Section 5 focus on the theory of reactive systems, especially *bigraphical reactive systems* (BRSs), and recast these in the proposed setting recovering crucial and useful constructions and properties such as relative (RPO) and idem (IPO) pushout which allows the synthesis of *labeled transition systems* (LTS) whose labels are minimal contexts. Finally, we will show that these constructions are respected by monoidal sortings (weak monoidal predicates).

Conclusions and final remarks are in Section 6. We assume the reader familiar with the basic definitions of bigraphs; see e.g. [25].

2 Supported precategories and presheaves

In this section we recall the notion of supported precategories (Definition 2.2), also known as s-(pre)categories [25], and propose a new generalization that overcome the limitation of supports. In the original notion, supports are intended to be subsets of a given set (hence the appellative of s-categories) or, *e.g.* in [27], objects of the category of finite sets. In the wake of the insights offered by profunctors (see Definition 2.4) we propose a presheaf category inspired to them as a context to model and study our generalization of supported (pre)categories. In particular, their relations and interactions. Profunctors are sometimes called (bi)modules or distrubutors. The interested reader may refer to [2,7] for an introduction to profunctor theory and to [1, 7, 29, 32] for some recent and interesting applications.

The concept of precategory relaxes a fundamental requirement of any category: the existence of the composition of each pair of compatible arrows. This choice is supported by various applications of precategories, especially in the theory of reactive systems [19] where, in the form of supported precategories led to important results such that the bigraphical framework [18, 22, 25] and its ability to synthesize minimal labeled transition systems.

Definition 2.1 (Precategory)**.** *A* precategory ${}^{\backprime}\mathrm{C}$ *consists of the same data as a category. The composition operator* $(- \circ -)$ *is, however, a partial function which satisfies*

(1) *for any arrow* $f : A \to B$*, both* $\mathrm{Id}_B \circ f$ *and* $f \circ \mathrm{Id}_A$ *are defined and equals to* f*;*

(2) *for any* $f : A \to B$*,* $g : B \to C$*,* $h : C \to D$*,* $(h \circ g) \circ f$ *is defined iff* $h \circ (g \circ f)$ *does and then* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

Definition 2.2 (Well supported (pre)category)**.** *Let* SET_f *be the category of finite sets. A* well supported category *is a pair* $(\mathrm{A}, |-|)$*, where* A *is a category and* $|-| : \mathrm{A}_1 \to \mathrm{SET}_f$ *is the so called* support function *such that for any* $g \circ f \in \mathrm{A}_1$*,* $|g \circ f| = |g| \cup |f|$*, and for any* $A \in \mathrm{A}_0$ $|\mathrm{Id}_A| = \emptyset$*. A* well supported precategory *is a pair* $({}^{\backprime}\mathrm{A}, | - |)$*, where* ${}^{\backprime}\mathrm{A}$ *is a precategory such that* $g \circ f$ *is defined iff* $|g| \cap |f| = \emptyset$*.*

Although SET_f has been used to model support so far, what is really used of this category is the monoidal structure given by set union. In the case of well supported precategories, composition is subject to the requirement of support disjointness that is a predicate over pair of finite sets i.e. supports. This may lead to wonder about generalizing beyond finite sets. For instance, supports may be strings and composition be disciplined by a grammar relating the structure of a (pre)category to a language. This observation led us to introduce the notion of *monoidal supported (pre)categories* where the term monoidal refers to the monoidal structure of supports.

Definition 2.3 (Monoidal supported (pre)category)**.** *Let* $(\mathrm{M}, \otimes, \epsilon)$ *be a small monoidal category. A* Monoidal supported category (over M or just M-supported) *is a pair* $(\mathrm{A}, |-|)$*, where* A *is a category and* $| - | : \mathrm{A}_1 \to \mathrm{M}$ *is the support function such that for any* $g \circ f$ $|g \circ f| = |g| \otimes |f|$ *and for any* $A \in \mathrm{A}_0$ $|\mathrm{Id}_A| = \epsilon$*. A* monoidal supported precategory *is a triple* $({}^{\backprime}\mathrm{A}, | - |, \varphi)$*, where* φ *is a predicate over* $\mathrm{M} \times \mathrm{M}$ *and* ${}^{\backprime}\mathrm{A}$ *is a precategory such that* $g \circ f$ *is defined iff* $\varphi(|g|, |f|)$*. If the application of* φ *to* $(\mathrm{A}, | - |)$ *determines* $({}^{\backprime}\mathrm{A}, | - |, \varphi)$ *then the first is said to be the* companion *of the latter.*

As anticipated, monoidal supported precategories are a generalization of well supported ones: every well supported precategory $({}^{\backprime}\mathrm{A}, | - |)$ is a monoidal supported precategory over $(\mathrm{SET}_f, \cup, \emptyset)$ with respect to the disjointness predicate "$| - | \cap | - | = \emptyset$" which can be easily expressed in the internal language of the support category SET_f (and in the internal logic of the presheaf category over supports).

M-supported categories owns full citizenship as categories since identities, composition and associativity are well defined thanks to the monoidal structure formed by supports. Moreover, Definition 2.3 can be further and easily extended taking into account the structure of M to define 2-cells (*e.g.* to capture support translations) and so on.

Definition 2.4 (The bicategory of profunctors)**.** *A profunctor* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *from the small category* C *to* D *is a functor* $\mathrm{D}^{op} \times \mathrm{C} \to \mathrm{SET}$*. Profunctors are morphisms of the bicategory* PROF *(sometimes also denoted as* MOD *or* DIST*) which has small categories as objects and natural transformation as 2-arrows. Given two profunctors* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *and* $G : \mathrm{D} \nrightarrow \mathrm{E}$*; their composite* $G \circ F : \mathrm{C} \nrightarrow \mathrm{E}$ *is defined as* $Lan_{\mathcal{Y}_{\mathrm{D}}} \widehat{G}\widehat{F}$ *where* $\widehat{F} : \mathrm{C} \to \mathrm{SET}^{\mathrm{D}^{op}}$ *and* $\widehat{G} : \mathrm{D} \to \mathrm{SET}^{\mathrm{E}^{op}}$ *are given by the cartesian closure of* CAT*.*

Profunctors are composed by using a coend to "trace out" the middle variable. Specifically given F and G as above $G \circ F$ is defined by the diagram

$$\begin{tikzcd}
\mathrm{D} \arrow[r, "\mathcal{Y}_{\mathrm{D}}"] \arrow[d, "\widehat{G}"'] & \mathrm{SET}^{\mathrm{D}^{op}} \arrow[dl, "Lan_{\mathcal{Y}_{\mathrm{D}}}\widehat{G}"] & \mathrm{C} \arrow[l, "\widehat{F}"'] \\
\mathrm{SET}^{\mathrm{E}^{op}} & &
\end{tikzcd}$$

and, unfolding the left Kan extension, the composite can be formulated as:

$$G \circ F \triangleq \int^{D \in \mathrm{D}} G(-, D) \times F(D, -) \cong \left(\coprod_{D \in \mathrm{D}} G_{(E,D)} \times F_{(D,C)} \right)_{/\approx} \tag{1}$$

where $\approx$ is the equivalence relation defined as

$$(g, f) \approx (g', f') \iff \exists d \in \mathrm{D}(D, D') : F_{(d, \mathrm{Id}_C)}(f') = f \wedge G_{(\mathrm{Id}_E, d)}(g) = g'.$$

Composition of profunctors is associative only up to isomorphism because of the product not being strictly associative in SET.

Automorphism over a small category C form the bicategory PROF(C). Its hom-category $[\mathrm{C}^{op} \times \mathrm{C}, \mathrm{SET}]$ is endowed with a monoidal structure induced by profunctor composition. In particular, the monoidal multiplication is given by the bifunctor $(- \circ -)$ and the monoidal unit is the identity profunctor Id_{C} i.e. the hom functor of C. Consider a monoid (A, μ, η) of PROF(C); the profunctor $A \circ A$ is isomorph, for each stage $(X, Z) \in \mathrm{C}^{op} \times \mathrm{C}$, to the set

$$\{(g, f) \mid g \in A(Y, Z) \wedge f \in A(X, Y) \wedge Y \in \mathrm{C}\}_{/\approx}$$

where $\approx$ is the equivalence relation of (1). This set seems to describe all pairs of composible arrows of some category. If sets generated by the presheaf A are thought as homsets, the monoidal unit determines which elements are identities and the monoidal multiplication can be interpreted as some notion of composition. A composition up to morphisms in C. Therefore, if C is discrete, every monoid in the hom-category of PROF(C) can be read as a category with the "same" objects as C since $\approx$ would be the equality relation.

In this perspective is became possible to either tweak the structure of index category to match the desired notion of composition for objects described by the presheaves considered, to use the notion of composition induced by the category as it is or both. This enables a great degree of freedom without the loss of important tools and results offered by a categorical approach. The key point is to intend presheaves as "generators" and categorical constructions over them as "constraints" or "queries" as briefly illustrated in the following example which is rather simple and straightforward but similar construction are able to capture more complex concepts such as relative and idem pushouts (*cf.* Section 5).

Example 2.5. *Let* (H, μ, η) *be a monoid of* $([\mathrm{C}^{op} \times \mathrm{C}, \mathrm{SET}], \circ, \mathrm{Id}_{\mathrm{C}})$*. Then the kernel pair of* $\mu : H \circ H \to H$ *describes every commuting square in the category modeled* H*.*

A precategory ${}^{\backprime}\mathrm{A}$ can be thought as a refinement of a suitable category, say A, since the homsets of the first are subsets of the ones of the latter. Let A be modeled by the presheaf monoid (H, μ, η). Then ${}^{\backprime}\mathrm{A}$ defines a subpresheaf of H whose characteristic map determines a predicate over the homsets of A such that it recovers ${}^{\backprime}\mathrm{A}$ from A.

This idea applies also to supported categories and precategories in order to formalize and study their relations within the same category. To capture the central rôle played by supports, profunctors have to be equipped with the monoid of supports leading to the definition of *monoidal supported profunctors*.

Definition 2.6 (The bicategory of monoidal supported profunctors). *Let* $(\mathrm{M}^{op}, \otimes, \epsilon)$ *be a small monoidal category.* M-$\textsc{Prof}$*, the bicategory of* M-supported profunctors*, has small categories as objects and presheaves of* $[\mathrm{M} \times \mathrm{D}^{op} \times \mathrm{C}, \textsc{Set}]$ *as morphisms from* C *to* D*. Composition is defined combining profunctor composition* (1) *with the lifting of* $(\mathrm{M}^{op}, \otimes, \epsilon)$ *through Day's convolution [9] (see diagram below). Specifically, given* $F : \mathrm{C} \nrightarrow_{\mathrm{M}} \mathrm{D}$ *and* $G : \mathrm{D} \nrightarrow_{\mathrm{M}} \mathrm{E}$*, their composite is defined as:*

$$(G \circ F) \triangleq Lan_{Lan_{\mathcal{Y}}\widehat{G}\widehat{F}}\mathcal{Y}(- \otimes -) \tag{2}$$

where $\widehat{F} : \mathrm{M} \times \mathrm{C} \to \textsc{Set}^{\mathrm{D}^{op}}$ *and* $\widehat{G} : \mathrm{D} \to \textsc{Set}^{\mathrm{M} \times \mathrm{E}^{op}}$ *exists by cartesian closure of* $\textsc{Cat}$.

$$\begin{tikzcd}
\mathrm{D} \arrow[r, "\mathcal{Y}"] \arrow[d, "\widehat{G}"'] & \textsc{Set}^{\mathrm{D}^{op}} \arrow[dl, "Lan_{\mathcal{Y}}\widehat{G}"] & \mathrm{M} \times \mathrm{C} \arrow[l, "\widehat{F}"'] \\
\textsc{Set}^{\mathrm{M} \times \mathrm{E}^{op}} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{M}^{op} \times \mathrm{M}^{op} \arrow[r, "Lan_{\mathcal{Y}}\widehat{G}\widehat{F}"] \arrow[d, "\mathcal{Y}(- \otimes -)"'] & \textsc{Set}^{\mathrm{E}^{op} \times \mathrm{C}} \arrow[dl, "Lan_{Lan_{\mathcal{Y}}\widehat{G}\widehat{F}}\mathcal{Y}(- \otimes -)"] \\
\textsc{Set}^{\mathrm{M}} &
\end{tikzcd}$$

In particular, ordinary profunctors are 1-supported. In the following we may refer to M-supported profunctors simply as "profunctors", when there is no risk of confusion. Notice that supported profunctor composition is associative only up to isomorphisms. In fact, unfolding (2) we have:

$$(G \circ F)_{(Q,E,C)} \cong \left(\coprod\nolimits_{(M,N),D} \mathrm{M}(M \otimes N, Q) \times F_{(N,D,C)} \times G_{(M,E,D)}\right)_{/\approx}$$

where $\approx$ is the equivalence relation such that:

$$\begin{aligned}
(q : M \otimes N \to Q, f, g) \approx (q' : M' \otimes N' \to Q, f', g') \iff& \\
&\exists m \in \mathrm{M}(M, M') \exists n \in \mathrm{M}(N, N') \exists d \in \mathrm{D}(D, D') : \\
&f = F_{(\mathrm{Id}_N, d, \mathrm{Id}_C)}(f') \wedge g' = G_{(\mathrm{Id}_M, \mathrm{Id}_E, d)}(g) \wedge q' = \mathrm{M}(m \otimes n, Q)(q).
\end{aligned}$$

Likewise $\textsc{Prof}(\mathrm{C})$, automorphisms over $\mathrm{C} \in \mathrm{M}$-$\textsc{Prof}$ form a bicategory on their own: M-$\textsc{Prof}(\mathrm{C})$.

Lemma 2.7. *For any small category* C*, automorphisms over* $\mathrm{C} \in \mathrm{M}$-$\textsc{Prof}$ *form the bicategory* M-$\textsc{Prof}(\mathrm{C})$ *whose hom-category* $[\mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}]$ *is monoidal.*

Proof. The hom-category of M-$\textsc{Prof}(\mathrm{C})$ is the presheaf category $[\mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}]$. This category is endowed with a monoidal structure whose multiplication follows from (2) and whose unit is the identity M-profunctor $\mathrm{Id}_{\mathrm{C}} = \mathrm{M}(-, \epsilon) \times \mathrm{C}(-, -)$. Associator, left and right unitors are defined by composition in M-$\textsc{Prof}(\mathrm{C})$. Then coherence conditions are straightforward. □

Examples 2.8. *Let* M *and* C *be a grupoidal category (i.e. every morphism has an inverse) and a discrete category respectively. Let* (A, μ, η) *be a monoid of the hom-category of* M-$\textsc{Prof}(\mathrm{C})$*. Then the notion of composition induced by the monoidal multiplication is up to arrows in* M*, that are isomorphisms between supports. Specifically,* $\approx$ *equates* $(q : M \otimes N \to Q, f, g)$ *to every* $(q' : M' \otimes N' \to Q, f', g')$ *such that* $M \cong M'$ *and* $N \cong N'$*. If* C *is a grupoidal category too, then the resulting composition is also up to isomorphism in the middle object and the monoid* A *can be interpreted as a category with the objects of the skeleton of* C *(i.e. equivalence classes of objects of* C *closed under isomorphisms).*

As an immediate consequence of the notion of companion of an M-supported precategory, also these can be modeled in a suitable hom-category of M-supported profunctors. The result, stated in the following proposition, reproduce the relation between an M-supported precategory and its companion leveraging the information of the predicate over supports associated to the precategory in order to "restrict" the monoid that describes its companion category characterizing a suitable partial monoid.

Proposition 2.9. *Every supported precategory can be modeled in a hom-category of* M-PROF *as a partial monoid.*

Proof. Let $(`\mathrm{A}, |-|, \varphi)$, $(\mathrm{A}, |-|)$ and (A, μ, η) be a M-supported precategory, its companion and the monoid describing the latter respectively. The predicate φ over $\mathrm{M} \times \mathrm{M}$ readily extends to a predicate over $A \circ A$ *i.e.* a morphism to the subobject classifier of the presehaf topos. This arrow determines a subobject of $A \circ A$ isomorph, for each stage (M, X, Z), to the set of triples of the like of $(m : |g| \otimes |f| = M, g : Y \to Z, f : X \to Y)$ where $f, g \in \mathrm{A}$ and $\psi(|g|, |f|)$ – if M is not discrete, this set would be of equivalence classes closed under morphisms of M *e.g.* support translations. $\square$

As a corollary of Proposition 2.9, the predicate over support pairs associated to each supported precategory has to be expressible in the internal language of the presheaf category (which is a topos). This fact naturally suggests to use this internal language as the formal mean for specifying these predicates. This issue was intentionally left out in Definition 2.3 for the sake of simplicity.

Product of supported profunctors Supported precategories were proposed as a formalization of bigraphs in which objects are interfaces, morphisms are bigraphs and supports are internal names [25, Def. 2.17]. In this model, arrow composition captures bigraph composition along a shared interface and is defined only if components do not share any internal name *i.e.* their supports are disjoints. Besides composition, bigraphs offer an other primitive operation to juxtapose them. This operation is defined only when operands do not share any internal name and is modeled endowing the precategory of bigraphs with a partial tensor, that is a tensor defined only upon suitable pairs of objects and pairs of arrows having domains and codomains for which the tensor is defined and disjoint supports. Then, the usual coherence condition for monoidal category are required to hold where the tensor is defined. In the case of bigraphical precategories, also coherence conditions of symmetric monoidal categories have to be satisfied, up to tensor definition [25, Def. 2.10].

This is captured in M-PROF(C) taking C to be a (symmetric) monoidal category.

Lemma 2.10. *Let* $(\mathrm{C}, \otimes, \epsilon)$ *be a monoidal category. Then the hom-category of* M-PROF(C) *presents a second monoidal structure beside the one induced by* M*-profunctor composition.*

Proof. The monoidal structure of C can be combined by product with the one of M rendering $M^{op} \times \mathrm{C} \times \mathrm{C}^{op}$ a monoidal category. This structure lifts to the hom-category of M-PROF(C) by straightforward application of Day convolution. $\square$

We usually use for monoidal structures lifted through Day convolution the same symbol of the lifted one when confusion seems unlikely, otherwise we rely on subscripting the intended category.

If this additional monoidal structure induces a monoid (H, μ, η) over a presheaf describing a supported category $(\mathrm{A}, |-|)$, then $\eta : \mathcal{Y}\epsilon \to H$ and $\mu : H \otimes H \to H$ describe a monoidal structure over A (defining unit and multiplication respectively). A has by construction the "same" objects of C (up to arrows in C) inheriting the data of $(\mathrm{C}, \otimes, \epsilon)$ for what concern objects. On the other hand, arrows depend only on information contained in H and are subject to the structure of $(\mathrm{M}, \otimes, \epsilon)$ which is lifted to the presheaf category by Day convolution:

$$H \otimes H = \int^{(M,V,X),(N,W,Y) \in \mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}} (H \times H)((M,V,X),(N,W,Y)) \\ \times \mathrm{M}(M \otimes N, -) \times \mathrm{C}^{op}(V \otimes W, -) \times \mathrm{C}(X \otimes Y, -).$$

For any stage (Q,S,T), $(H \otimes H)_{(Q,S,T)}$ is isomorph to the set of tuples (modulo the equivalence of the coend) like (h, h', m, c, c') and such that $h \in H_{(M,V,X)}$, $h' \in H_{(N,W,Y)}$, $m : M \otimes N \to Q$, $c : S \to V \otimes W$ and $c' : X \otimes Y \to T$. These tuples are mapped by the natural transformation μ in elements of the set described by H at the same stage (Q,S,T). This tensor is well defined since both $(\mathrm{M}, \otimes, \epsilon)$ and $(\mathrm{C}, \otimes, \epsilon)$ do.

The (symmetric) monoidal structure of an M-supported category $(\mathrm{A}, |-|)$ offers the base to define a partial (symmetric) monoidal structure on every precategory whose companion is A. The predicate over support pairs of $({}^{\backprime}\mathrm{A}, |-|, \varphi)$ restricts the monoidal multiplication of A on the base of the supports of the arrows involved. In [25, Def. 2.10] the tensor of a partial monoidal precategory can be partialy defined on both arrows and objects (albeit some additional coherence conditions are required for the symmetric ones). The partiality on objects is independent from supports and vice versa. In order to define the partial monoidal structure over ${}^{\backprime}\mathrm{A}$ this information is encoded in a predicate over pairs of objects which (in combination with the one over supports) restricts the monoidal multiplication of A. Likewise of Proposition 2.9, partial monoidal M-supported precategories can be easily modeled in M-Prof.

Free algebras over monoidal structures An important application of presheaf categories is to allow to define as free algebras also languages with complex operators such as binders, non-interference parallels, substitutions, etc., by taking advantage of specific tensor products (and their adjoints) [6,11,12,20,21,30]. A similar result holds in our settings: the two monoidal structures (say $\circ$ and $\bullet$) of the hom-category M-Prof(C) yield two orthogonal ways to compose presheaves offering a natural way to express "support-aware" operations on the elements described by these presheaves. Moreover, this setting can be used to define languages based on such operations as initial algebras for suitable endofunctors arising from these tensors (*cf.* Section 3). This class of $\bullet\circ$-terms is an instance of the general notion captured by the following definition.

Definition 2.11 (Free $\mathcal{M}$-algebras). *Let* C *be a small category with a collection of monoidal structures* $\mathcal{M}$*. For* X *an object of* C*, we define* $T_{\mathcal{M}}(X)$*, the* free $\mathcal{M}$-algebra *over* X*, as the free* $\Sigma_{\mathcal{M}}$*-algebra over* X*, where* $\Sigma_{\mathcal{M}}(Y) = \coprod_{\star \in \mathcal{M}} Y \star Y$.

Let Y be a presheaf and $T_{\bullet\circ}(Y)$ be the free $\bullet\circ$-algebra, and let H be a monoid for both $\bullet$ and $\circ$. Then, the two multiplications induce an algebraic structure over H, and hence by initiality there exists $\nu : T_{\bullet\circ}(Y) \to H$. This defines an interpretation of $\bullet\circ$-terms over Y into elements of H, and ultimately in the supported category A modeled by H. If ν is epic, then the language defined by $T_{\bullet\circ}(Y)$ is a complete representation for A. On the other hand, every mono $H \to T_{\bullet\circ}(Y)$ can be seen as the definition of a *normal form.* These constructions readily extend to morphisms between the monoids induced by $\circ$ and $\bullet$, lifting normal forms and interpretations to the category of monoids.

3 An abstract presentation of bigraphs

In this section we apply the results of the previous to provide a presentation of bigraphs in suitable presheaf categories. Taking advantage of this rich setting, we describe bigraphs, their components (*i.e.* signatures, link and place graphs), how these combine yielding bigraphical structures and how relate to terms and algebras used to define them [24]. All of this, within the same environment category.

We focus on a new flavor of bigraphs introducing *bigraphs with abstract names* and operations with "built-in" renaming. This feature renders these operation total allowing to modeled this kind of bigraph by a category instead of a precategory as in the case of concrete bigraphs. This characterization, in a sense lies, between Milner's concrete and abstract bigraphs.

Our presentation relay on the category of finite ordinals and bijective permutations $\mathbb{B}$ to model abstract names and remaings but (as will be shown at the end of the section) Milner's concrete bigraphs are readily recovered by means of $\textsc{Set}_b$ – the category of finite sets and bijective functions, whose skeleton is $\mathbb{B}$. Moreover, their companion supported category will define concrete bigraphs with built-in renaming.

3.1 Signatures and control maps

A *basic signature* is a set of controls, each of which is assigned a finite number called *arity* [25, Def. 1.1]. Therefore, a signature K is a list $(K_1, \dots, K_n)$ where K_i is the arity of the i-th control. Hence we can define the category $\mathcal{K}$ of *basic signatures* as $\mathcal{K} \triangleq \mathbb{N}^*$, *i.e.* the monoidal closure of the discrete category of natural numbers (which is discrete too).

This notion of signature can be easily adapted to more expressive settings, as needed. In particular, *activity* of controls can be modeled by taking $\mathcal{K} = (2 \times \mathbb{N})^*$; binding signatures are modeled by $(\mathbb{N} \times \mathbb{B})^*$; directed signatures are objects of $(\mathbb{N} \times \mathbb{N}^{op})^*$ – $\mathbb{N}$ is self dual, but we stress this definition to highlight the polarity of links, names and ports – *etc.*

Given a signature K and a graph with nodes in V, a *control map* associates to each node in V a control in K. V can be seen as a set of names and can be represented by an object of $\mathbb{B}$. Arrows of $\mathbb{B}$ are finite permutations and "hide" the order implicit in the objects of $\mathbb{B}$. In particular, $n \in \mathbb{B}$ represents a set $\{x_1, \dots x_n\}$ of n distinct names, thus abstracting from any particular name (*cf.* $\textsc{Set}_b$).

We define the category of *control maps over a signature* K as the comma category

$$\mathcal{C}_K \triangleq I_{\mathbb{B}} \downarrow U_{\mathcal{K}} K$$

where $I_{\mathbb{B}} : \mathbb{B} \to \mathbb{F}$ is the inclusion functor from $\mathbb{B}$ to $\mathbb{F}$, the category of finite ordinals and functions; $U_{\mathcal{K}} : \mathcal{K} \to \mathbb{F}$ is the "length" forgetful functor, and $K : 1 \to \mathcal{K}$ is the constant functor returning the signature K. Objects of $\mathcal{C}_K$ are (isomorphic to) control maps $c : V \to K$, and a morphism $\pi : c \to c'$ amounts to a bijection $\pi : \mathrm{dom}(c) \to \mathrm{dom}(c')$ respecting controls; these correspond precisely to the notion of *support translation* [25, Def. 2.13].

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "c"'] & & V \arrow[dl, "c'"] \\ & K & \end{tikzcd}$$

Thus the sum in $\mathbb{B}$ extends to $\mathcal{C}_K$, rendering $(\mathcal{C}_K, \oplus, \varnothing)$ a symmetric monoidal category where multiplication and unit represent disjoint union of maps (because of names being abstract there is an implicit rename) and the empty control map $\varnothing : 0 \to K$, respectively.

3.2 Place graphs

Definition 3.1 ([25, Def. 2.1])**.** *A* concrete place graph $P = (V_P, ctrl_P, prnt_P) : m \to n$ *over a given signature K [25, Def. 2.1] is a triple having an inner interface m and an outer n, both finite ordinals. These index respectively* sites *and* roots *of the place graph. P has a finite set of nodes V_P and the control map $ctrl_P : V_P \to K$ assigns a control in K to each of them. Nodes, sites and roots are organized in a forest described by the parent map $prnt_P$ and such that sites are leaves and roots are exactly n.*

A concrete place graph, and in general a place graph, is an object which exposes three distinct informations: the inner interface, the outer interface, and the support. The support is usually intended as the set of nodes of a place graph (resp. link graph or bigraph) but support translations for place graphs (resp. link graph or bigraph) are defined in [25, Def. 2.4] to be renaming of the internal names that respect controls. Therefore, not only names have to be observable, but also controls have to be so. This suggests to take the control map as the support. These three informations are exactly what is needed for defining composition and juxtaposition of place graphs and thus can be seen as its *type*.

Definition 3.2 (place graph with abstract names)**.** *A place graph with abstract names over a given signature K is a concrete place graph except for the set nodes being a finite ordinal drawn from $\mathbb{B}$ and the control map which is a suitable object of $\mathcal{C}_K$.*

Likewise concrete place graphs on a given signature K are modeled by supported morphism, those with abstract names are modeled as arrows between place graph interfaces *i.e.* finite ordinals. These form the *category of place graphs with abstract names* $\textsc{Pga}_K$ which is a $\mathcal{C}_K$-supported category and, by the results presented in the previous section, is captured by a monoid in the hom-category of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{N})$ *i.e.* the presheaf category $[\mathcal{P}_K, \textsc{Set}]$ where

$$\mathcal{P}_K \triangleq \mathcal{C}_K \times \mathbb{N}^{op} \times \mathbb{N}.$$

The sum of finite ordinals endows $\mathbb{N}$ with a symmetric monoidal structure with 0 as the unit. This is lifted to presheaves by application of Lemma 2.10 rendering $([\mathcal{P}_K, \textsc{Set}], \oplus, \mathcal{Y}\varepsilon)$ a symmetric monoidal category (and the Yoneda embedding a monoidal functor) where ε is a short end for $(\varnothing, 0, 0)$ *i.e.* the type of the empty place graph, and the tensor obtained through the Day's convolution is defined as follows:

$$(A \oplus B)_P = \int^{P_1, P_2 \in \mathcal{P}_K} A_{P_1} \times B_{P_2} \times \mathcal{P}_K(P_1 \oplus P_2, P). \tag{3}$$

For any stage $P \in \mathcal{P}_K$, $(A \oplus B)_P$ is isomorph to the set of classes of triples closed under arrows of $\mathcal{C}_K$ ($\mathbb{N}$ is discrete) *i.e.* support translations. These triples are of the like of (a, b, p) where $a \in A_{P_1}$, $b \in B_{P_2}$ and $p : P_1 \otimes P_2 \cong P$ is a renaming that allows to obtain P as the sum of P_1 and P_2. Therefore this monoidal structure would express "support-aware" juxtaposition. In particular, the monoidal multiplication induced by this structure over the presheaf describing the arrows of $\textsc{Pga}_K$, would define a tensor over the category $\textsc{Pga}_K$ that captures juxtaposition of place graphs with abstract names as a total operation thanks to the built-in rename.

Composition of place graphs with abstract names is modeled in $\textsc{Pga}_K$ as the composition of morphisms. This is captured (up-to support translation) by the monoidal structure induced by profunctor composition over $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{N})$. Seeking adherence with the usual ordering for arrow composition, we introduce the tensor $(- \odot -)$:

$$(A \odot B)_{(C,M,N)} \triangleq \int^{C_1,\, C_2 \in \mathcal{C}_K} \mathcal{C}_K(C_1 \oplus C_2, C) \times \int^{Q \in \mathbb{N}} A_{(C_2,Q,N)} \times B_{(C_1,M,Q)} \tag{4}$$

whose arguments are swapped respect to the tensor $(- \circ -)$ induced by profunctor composition. The monoidal unit still is the identity profunctor on $\mathbb{N}$:

$$I_{K(C,M,N)} \triangleq \mathcal{C}_K(C, \varnothing) \times \mathbb{N}(M, N).$$

Therefore, $([\mathcal{P}_K, \textsc{Set}], \odot, I_K)$ is a monoidal category and this structure would express composition up-to support translations.

These two monoidal structures yield two orthogonal ways for composing objects in $[\mathcal{P}_K, \textsc{Set}]$ and allows to express "support-aware" operations on these objects and the elements they describe. In particular, these induces two monoidal structures above the presheaf describing place graphs with abstract names.

Place graphs with abstract names can be seen as freely generated by *placings* and *ions*. Therefore we can define a term language for describing place graph as the free $\oplus\odot$-algebra over a presheaf describing palcings and ions thanks to Definition 2.11. Placings are characterized by the presheaf

$$plc_{K(c,M,N)} \triangleq \mathcal{C}_K(c, \varnothing) \times \mathbb{F}(M, N)$$

and ions by

$$ion_{K(c,M,N)} \triangleq \coprod_{k:1\to K} \mathcal{C}_K(c, k) \times (\mathbb{N}(M, 1) + \mathbb{N}(1, N)).$$

When $c \cong \varnothing$, the first yields precisely all functions from the inner interface M to the outer one N *i.e.* forests without controls. When c is a map which sends the only node in its domain to the control k, the second yields all pairs of functions in $\mathbb{N}$ from the inner face to the only node and from this node to the outer face, that is the parent map of a ion with control k. Then the presheaf of $\oplus\odot$-terms describing place graphs with abstract names over K can be defined as the free $\oplus\odot$-algebra:

$$Plc_K \triangleq T_{\oplus\odot}(plc_K + ion_K).$$

Grafting defines an epimorphism from Plc_K to $\textsc{Pga}_K$ and every mono on the opposite direction determines a normal form for the $\oplus\odot$-terms described by Plc_K.

Example 3.3 (Place graph with sharing)**.** *Bigraphs with sharing [5] (here with abstract names) are defined by merging pure link graphs with place graphs with sharings which rely for their underling structure on directed acyclic graphs instead of forests. Hence placings are set-valued functions never returning the empty set which are described by*

$$s\text{-}plc_{K(C,M,N)} \triangleq \mathcal{C}_K(C, \varnothing) \times \textsc{Set}(J_M, \wp^+(J_N))$$

where $J : \mathbb{N} \rightarrowtail \textsc{Set}$ is the obvious injection functor. Ions are the same of pure place graphs. $T_{\oplus\odot}(s\text{-}plc_K + ion_K)$ defines the presheaf of terms.

3.3 Link graphs

Definition 3.4 ([25, Def. 2.2])**.** *A concrete link graph $L = (V_L, E_L, ctrl_L, link_L) : X \to Y$ over a given signature K is a quadruple having an inner interface X and an outer Y, both finite sets of names. L has a finite sets of nodes V_L and of edges E_L, a control map $ctrl_L : V_L \to K$ and a link map $link_L$ from inner names of X and ports (i.e. elements of the finite ordinal associated to each node by its control) to edges and outer names of Y.*

Following the structure of the previous section we introduce the category $\textsc{Lka}_K$ whose objects are finite ordinals and whose arrows are link graphs with abstract names. These can be readily defined taking finite ordinals instead of finite sets of names in [25, Def. 2.2]. For our purposes and consistently with the algebra for link graphs proposed by Milner's in [24] we safely omit the set of edges from the type of a link graph since they are more critical to the definition of the link map of concrete link graphs rather than the definition of operations over them.

Therefore, a link graph with abstract names over K exposes as a type a triple (C, X, Y), where the first (its control map) is an object of $\mathcal{C}_K$ and the remaining (its inner and outer names) are objects of $\mathbb{B}$. This suggests to model $\textsc{Lka}_K$ and the presheaf of terms describing its arrows in the hom-category of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{B})$ *i.e.* $[\mathcal{L}_K, \textsc{Set}]$ where

$$\mathcal{L}_K \triangleq \mathcal{C}_K \times \mathbb{B}^{op} \times \mathbb{B}.$$

$[\mathcal{L}_K, \textsc{Set}]$ presents two monoidal structures. The first, $([\mathcal{L}_K, \textsc{Set}], \odot, I_K)$, is induced by arrow composition in $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{B})$ and the second, $([\mathcal{L}_K, \textsc{Set}], \oplus, \mathcal{Y}\varepsilon)$, arises by application of Lemma 2.10 from the symmetrical monoidal structure of $\mathbb{B}$ defined by the sum of finite ordinals. These two structures – whose multiplications are defined as

$$(A \odot B)_{(C,M,N)} \triangleq \int^{C_1,\, C_2 \in \mathcal{C}_K} \mathcal{C}_K(C_1 \oplus C_2, C) \times \int^{Q \in \mathbb{B}} A_{(C_2,Q,N)} \times B_{(C_1,M,Q)} \tag{5}$$

and as

$$(A \oplus B)_L = \int^{L_1, L_2 \in \mathcal{L}_K} A_{L_1} \times B_{L_2} \times \mathcal{L}_K(L_1 \oplus L_2, L) \tag{6}$$

respectively – allows to model composition and juxtaposition up-to isomorphisms in the index category. In particular, these induce two monoidal structures over the presheaf of link graphs with abstract names, one captures arrow composition of $\textsc{Lka}_K$ and one models a tensor product and its unit.

Terms for describing link graphs with abstract names can be readily defined applying Definition 2.11 in $[\mathcal{L}_K, \textsc{Set}]$ and be freely generated from ions and wirings. Wirings are described by the following presheaf:

$$wir_{K(C,X,Y)} \triangleq \mathcal{C}_K(C, \varnothing) \times \mathbb{F}(X, Y+1).$$

When $C = \varnothing$, this yields precisely all partial functions from X to Y which are link graphs without nodes, *i.e.* wirings. Ions are determined by the presheaf:

$$ion_{K(C,X,Y)} \triangleq \coprod_{k:1 \to K} \mathcal{C}_K(C, k) \times \mathbb{B}(X, 0) \times \mathbb{B}(C(*), Y).$$

When C is a map that sends the only node of its domain in the control k this yields all bijection from the ports of the node to the outer interface *i.e.* a k-ion. Lnk_K, the presheaf of $\oplus\odot$-terms describing link graphs for a given signature K, can be defined as the free $\oplus\odot$-algebra in $[\mathcal{L}_K, \textsc{Set}]$ over the presheaf $wir_K + ion_K$:

$$Lnk_K \triangleq T_{\oplus\odot}(wir_K + ion_K).$$

3.4 Bigraphs

Likewise a *concrete bigraph* on a given signature K [25, Def. 2.3] is obtained by merging a concrete place graph and a concrete link graph ensuring that these structures share their control maps, a *bigraph with abstract names* is composed by a place graph and a link graph with abstract names and a common control map.

The category $\textsc{Bga}_K$ can be modeled in the hom-category of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{N}\times\mathbb{B})$ *i.e.* $[\mathcal{B}_K, \textsc{Set}]$ where

$$\mathcal{B}_K \triangleq \mathcal{C}_K \times (\mathbb{N} \times \mathbb{B})^{op} \times (\mathbb{N} \times \mathbb{B})$$

is determined by pulling back the forgetful functor from $\mathcal{P}_K$ to $\mathcal{C}_K$ along the one from $\mathcal{L}_K$ as shown in (7).

$$\begin{tikzcd} \mathcal{B}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_K \arrow[d] \\ \mathcal{L}_K \arrow[r] & \mathcal{C}_K \end{tikzcd} \tag{7}$$

The presheaf category $[\mathcal{B}_K, \textsc{Set}]$ presents two monoidal structures: $([\mathcal{B}_K, \textsc{Set}], \odot, I_K)$ and $([\mathcal{B}_K, \textsc{Set}], \oplus, \mathcal{Y}\varepsilon)$. The first arise from profunctor composition and the second from finite ordinal sum. Both can be – consistently with the definition of bigraphs and their category – projected through (7) to recover their counterpart: (4) and (3) in

$[\mathcal{P}_K, \textsc{Set}]$ and (5) and (6) in $[\mathcal{L}_K, \textsc{Set}]$ respectively. Moreover, (7) offers an alternative and consistent definition of the symmetric monoidal structure on $\mathcal{B}_K$ which results by "merging" the monoidal structures of $\mathcal{P}_K$ and $\mathcal{L}_K$: the tensor $\oplus$ is the unique map given by the universal property of $\mathcal{B}_K$.

Terms for describing link graphs with abstract names can be readily defined applying Definition 2.11 in $[\mathcal{B}_K, \textsc{Set}]$ and be freely generated from placings, wirings and ions. Ions are the contact surface between those orthogonal structures, because they involve controls (which are shared), while elementary placings and linkings do not share any data and therefore are basically their counterparts in $[\mathcal{P}_K, \textsc{Set}]$ and $[\mathcal{L}_K, \textsc{Set}]$. In particular, they correspond to them when composed to the injection functors from $\mathcal{P}_K$ and $\mathcal{L}_K$ to $\mathcal{B}_K$ that fill the missing information with the the units of $\mathbb{B}$ and $\mathbb{N}$ respectively. This applies also to bigraphical ions which are defined by the presheaf:

$$ion_{K(C,(M,X),(N,Y))} \triangleq \coprod_{k:1\to K} \mathcal{C}_K(C,k) \times \mathbb{N}(M,1) \times \mathbb{N}(1,N) \times \mathbb{B}(X,0) \times \mathbb{B}(C(*),Y).$$

This yields all bigraphical k-ions when C is $* \mapsto k$, M, N are 1 and X is empty. This presheaf can be projected to the ion_K of $[\mathcal{L}_K, \textsc{Set}]$ and to the ion_K of $[\mathcal{P}_K, \textsc{Set}]$. Then we define Big_K, the presheaf of $\oplus\odot$-terms describing bigraphs with abstract names over K, as the free $\oplus\odot$-algebra in $[\mathcal{B}_K, \textsc{Set}]$ over the presheaf $wir_K + plc_K + ion_K$:

$$Big_K \triangleq T_{\oplus\odot}(wir_K + plc_K + ion_K).$$

3.5 Concrete bigraphs

Well supported (pre)catetories, as in the case of Milner's concrete bigraphs [25, Def. 2.17], have their support modeled in $\textsc{Set}_f$. However, this category presents a very rich structure which is not used by supports and operations over them (*i.e.* union, translation and check for disjointness). As formalized [27, 28], support translation needs to form a grupoidal structure over supports. Therefore, supports are modeled in the category of finite sets and bijective functions $\textsc{Set}_b$ whose skeleton is $\mathbb{B}$. Set union endows this category with a symmetric monoidal structure whose unit is the empty set.

The precategory of Milner's concrete bigraphs $(\grave{}\textsc{Bg}_K, |-|, -\#-)$, where $(-\#-) \triangleq |-| \cap |-| = \emptyset$, is modeled in a suitable hom-category $\textsc{Set}_b$-$\textsc{Prof}$ as any well-supported (pre)category. Their companion category defines *concrete bigraphs with renamins* in which composition and juxtaposition are total operations thanks to built-in renaming. Terms algebras for describing these flavors of bigraphical languages can be readily defined applying Definition 2.11 also in this presheaf category. Similar characterizations can be easily formulated in the proposed settings for several variations on bigraphs such as, but not limited to, local bigraphs [4, 23], directed bigraphs [14–16], bigraphs with sharing [5] *etc.*

4 Sortings

In the theory of reactive systems a crucial part of the modelization of a language or a system is the definition of a sorting. In Milner's definition [25, Cap. 6] these are judgments over morphisms of precategories (*e.g.* bigraphs). The notion was refined by Debois with the introduction of *predicate sortings* which are based on the concept of *decomposable predicates* [3]. Predicate sortings are a subclass of Milner's ones but they suffice to capture the vast majority of practical scenarios as illustrated in [10, Cap. 6]. Both Milner's and Debois's notion of sortings can be captured in our settings as predicates over the presheaves modeling (pre)categories. In the present section we recast and generalize results of the field into our settings.

Monoids of a monoidal category $(\mathrm{C}, \otimes, I)$ form the category $\textsc{Mon}(\mathrm{C})$. An arrow $f : (X, \mu_X, \eta_X) \to (Y, \mu_Y, \eta_Y)$ of this category is given by a morphism $f : X \to Y \in \mathrm{C}$ such that $f \circ \eta_X = \eta_Y$ and $f \circ \mu_X = \mu_Y \circ (f \otimes f)$. If C is a topos and its monoidal structure is "well behaved" (we formalize this point in a moment) then the induced category of monoids results to be a topos allowing to take advantage of such a rich structure to study $\textsc{Mon}(\mathrm{C})$ and project its interesting properties on the objects modeled in it. The monoidal structure of C is said to be well behaved if it allows to lift products to monoids; reworded it distributes[1] over the cartesian product, *i.e.* there exists $\psi_{A,B,C,D} : (A \times B) \otimes (C \times D) \to (A \otimes C) \times (B \otimes D)$ natural in A, B, C, D.

Lemma 4.1. *Let* $(\mathrm{C}, \otimes, I)$ *be a monoidal category and let* $\textsc{Mon}(\mathrm{C})$ *be the category of monoids induced by its monoidal structure. If* C *is a topos and* $\otimes$ *distributes over* $\times$*, then also* $\textsc{Mon}(\mathrm{C})$ *is a topos.*

Proof. First we prove the existence of finite products, subobject classifier and then of power objects. The terminal object 1 is lifted from C to the monoid $(1, \mu_1, \eta_1)$ – multiplication and unit uniquely exist by terminality and trivially satisfy the monoidal axioms – which is terminal by terminality of 1. Binary products are lifted pointwise thanks of $\otimes$ distributing over products. Ω, the subobject classifier of C, is then endowed with a monoidal structure mimicking the definition of the logical and. The monoidal unit is $\tau \triangleq \top\circ! : I \to \Omega$ and the monoidal multiplication $\owedge : \Omega \otimes \Omega \to \Omega$ is defined as the characteristic map of the $m : A \rightarrowtail \Omega$ uniquely determined by the unique and minimal epi-mono factorization of $\top \otimes \top : 1 \otimes 1 \to \Omega \otimes \Omega$.

$$\begin{tikzcd}
1\otimes 1 \arrow[dr, dashed, "e"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "\top\otimes\top"'] & & \\
& A \arrow[r, "!"] \arrow[d, tail, "m"] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, "\top"'] \\
& \Omega\otimes\Omega \arrow[r, "\owedge"] & \Omega
\end{tikzcd}$$

Any morphism φ to $(\Omega, \owedge, \tau)$ has a pullback along the truth arrow from the terminal monoid. In particular, this describes (Y, μ_Y, η_Y), a sub monoid of the domain (X, μ_X, η_X) of φ. Y is determined pulling back φ along $\top$ in C; because of φ being monoidal and by definition of $\owedge$ (which can be seen as a "logic and for $\otimes$"), this extends to μ_X and η_X defining a pullback in $\textsc{Mon}(\mathrm{C})$. A similar construction can be used to lift power objects in presence of monoidal morphisms. □

Definition 4.2 (Monoidal Predicate)**.** *A morphism to the subobject classifier of* C *a topos with a monoidal structure that distributes over products, is said to be a* monoidal predicate *if it defines an arrow to the subobject classifier of the topos of monoids* $\textsc{Mon}(\mathrm{C})$.

Monoidal predicates characterize sub-monoids in a suitable topos. This applies also to hom categories like M-$\textsc{Prof}(\mathrm{C})$ – especially for the presheaf categories used in Section 3 – because $(- \circ -)$ distributes over $(- \times -)$ as stated by the following lemma.

Lemma 4.3. *Let* $(- \circ -)$ *denote the tensor product of the monoidal structure induced by supported profunctor composition over the hom-category of* M-$\textsc{Prof}(\mathrm{C})$*. There exists a natural transformation* $\psi_{A,B,C,D} : (A \times B) \circ (C \times D) \to (A \circ C) \times (B \circ D)$.

Proof. For any A, B, C and D of the presheaf topos $[\mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}]$, $(A \times B) \circ (C \times D)$ is a subobject of $(A \circ C) \times (B \circ D)$. □

[1] This distributive law between tensor products differs from the usual notion of distribution in bimonoidal categories.

Monoidal predicates define inclusions between structures described by monoids. In particular (*cf.* Examples 2.8) given a monoid that describes a category A, a monoidal predicate over it determines a subcategory B of A such that $\mathrm{A}_0 = \mathrm{B}_0$.

Example 4.4 (Bigraphs with and without sharing)**.** *Consider the presheaves Big_K and $S\text{-}Big_K$ describing $\odot\oplus$-terms characterizing bigraphs with abstract names and those with sharing respectively. Let $\varphi : S\text{-}Big_K \to \Omega$ be a predicate checking whatever a term describe a bigraph without sharing (it suffices to check whatever the term contains something from s-plc$_K$ but not from plc$_K$; a check that can be expressed in the internal language of the topos). Then φ characterize Big_K as a subpresheaf of $S\text{-}Big_K$. Moreover, φ is a monoidal predicate* w.r.t *$\odot$ and $\oplus$ since bigraphs described by the unit have identities as place graphs and juxtaposition and composition can not create any sharing. A similar results holds for the presheaves describing bigraphs (instead of terms) and between presehaves for terms and bigraphs:*

$$
\begin{tikzcd}
\text{S-}\textsc{Bga}_K & \textsc{Bga}_K \arrow[l, hook'] \\
S\text{-}Big_K \arrow[u, hook] & Big_K \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}
$$

corroborating claim of expressiveness for the proposed setting.

Although interesting, monoidal predicates fail short to capture relations between supported precategories (since these are modeled by partial monoids) or weaker relations which refine homsets but are not required to respect the definition of categories or precategories like sortings. However, monoidal predicates can be worked a bit, taking advantage of the internal language of the topos, to meet the requirement of predicates used to model supported precategories as illustrated by the following example.

Example 4.5. *Let the monoid (H, μ_H, η_H) model the companion $(\mathrm{A}, |-|)$ of a supported precategory $({}^{\backprime}\mathrm{A}, |-|, \psi)$. Given a monoidal predicate φ over H, the pullback of ψ along $\varphi \circ \mu = \varphi \owedge \varphi$ defines a sub-precategory of ${}^{\backprime}\mathrm{A}$ whose companion is the supported subcategory of A determined by φ.*

A decomposable predicate is a judgment P over morphisms of a given (pre)category and such that a positive judgment over an arrow implies a positive judgment over any of its components *i.e.* $P(g \circ f) \Rightarrow P(f) \wedge P(g)$. Predicates of this kind are strongly connected with the concept of (de)composition which we model as monoidal structures. This notion readily generalizes to predicates decomposable with respect to any given monoidal structure suggesting a generalization of Debois's results (in particular the ability to transfer RPOs). Because decomposable predicates are not required to define sub-(pre)categories – nothing forbids to use such a predicate to "ban" an identity *e.g.* to ban every bigraphs that factors through that interface – it is not hard to be convicted that monoidal predicates fail to capture and generalize decomposable ones. However, decomposable predicates can be recovered relaxing Definition 4.2 in a way that preserve the monoidal structure only up to suitable conditions that capture the notion of decomposability.

Definition 4.6 (Weak Monoidal Predicate)**.** *Let C be a topos with a monoidal structure $(\mathrm{C}, \otimes, I)$ satisfying Lemma 4.1 A morphism $\varphi : M \to \Omega$ is said to be a* weak monoidal predicate *if (M, μ, η) is a monoid and $\varphi \circ \mu = \varphi \owedge \varphi$.*

Likewise monoidal predicates, a predicate can be weakly monoidal with respect for several tensors: it just have to satisfy the above definition for each tensor separately.

Let φ be a weak monoidal predicate for M. φ and $\varphi \circ \mu = \varphi \owedge \varphi$ describe Y and X, subobjects of M and $M \otimes M$ respectively as illustrated by the diagram below.

$$
\begin{tikzcd}
M \otimes M \arrow[r, "\mu"] \arrow[d, "\varphi\otimes\varphi"'] & M \arrow[d, "\varphi"] \\
\Omega\otimes\Omega \arrow[r, "\owedge"'] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, dashed, "\nu"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
M \otimes M \arrow[r, "\mu"'] & M \arrow[r, "\varphi"'] & \Omega
\end{tikzcd}
$$

The unique arrow ν defined by the pullback Y can be seen as the restriction of the monoidal multiplication μ to the "information" of M that satisfies φ. If we take M to be an object of SET, X determines all the pairs of operands $a, b \in M$ such that if (with a little abuse of notation) $a \otimes b$ satisfies φ then also a and b do ($a \otimes b \in Y \Rightarrow a, b \in Y$).

Proposition 4.7 (Decomposable predicates). *Let P be a decomposable predicate on the morphisms of a small category* C *and let the monoid (H, μ_H, η_H) over the profunctor H describe* C *(cf. Examples 2.8). There exists a weak monoidal predicate $\varphi_P : H \to \Omega$, whose subobject is isomorphic, for any stage $(C, D) \in \mathrm{C}_0 \times \mathrm{C}_0$, to $\{f \in \mathrm{C}(C, D) \mid P(f)\}$.*

Proof. Let the profunctor X to be defined for each stage (C, D) as the set $\{f \in \mathrm{C}(C, D) \mid P(f)\}$ and define φ_P to be its characteristic map $\chi_X : H \to \Omega$. Because of P being decomposable ($p(g \circ f) \Rightarrow p(f) \wedge p(g)$) $\chi_X \circ \mu_H = \chi_X \owedge \chi_X$ (*cf.* proof of Lemma 4.1). □

Because of Proposition 4.7, weak monoidal predicates capture a class of objects strictly greater than those used to model supported precategories. Moreover, this result offers a context for a logic for decomposable predicates which is currently missing (albeit some results in this direction are present in [26]).

In [10, Ch. 5] the notion of decomposable predicates is extended to well supported precategories, in particular to bigraphical ones. Proposition 4.7 extends to these precategories, and in general to monoidal supported ones, with the only difference that have to be used partial monoids instead of total ones.

Proposition 2.9 offers a way to model a supported precategory $(`\mathrm{A}, |-|, \psi)$ taking advantage of the monoid (H, μ, η) modeling its companion category A. Arrow composition is described by a sort of restricted monoidal multiplication $\nu : X \to H$ whose domain is the subobject of $H \circ H$ induced by ψ (ν factorizes through μ into a mono). Then, the notions of monoidal and weak monoidal predicates can be translated to partial monoids and their representation based on "multiplication over subobjects" defining a class of predicates that captures and generalizes decomposable predicates for supported precategories.

4.1 Bigraphical monoidal sorting

Predicate sortings for reactive systems are based on decomposable predicates which are generalized by the notion of weak monoidal predicates as stated by Proposition 4.7. Therefore, we refer to weak monoidal predicates over monoids describing bigraphs as *bigraphical monoidal sortings*.

Example 4.8 (CCS). *Consider the signature Σ_{CCS} with controls $(0, 1, 1)$ or, for readability,* $(\mathsf{alt}{:}0, \mathsf{send}{:}1, \mathsf{get}{:}1)$ *given for the bigraphical encoding of the CCS proposed in [25, Def. 6.5]. In order to rule out bad formed bigraphs, we define – by means of the internal language of the boolean topos $[\mathcal{B}_{\Sigma_{CCS}}, \mathrm{SET}]$ – a predicate on the presheaf of bigraphs:*

$$\varphi_{(c,X,Y)}(G) = \bigwedge_{v \in \mathrm{dom}(c)} \mathrm{prnt}_G(v) \in \mathrm{dom}(c) \to (c(v) \in A \leftrightarrow \mathrm{prnt}_G(v) \notin A)$$

where $A = \{\mathsf{send}, \mathsf{get}\}$, $(c, X, Y) \in \mathcal{B}_{\Sigma_{CCS}}$, $G \in \mathrm{BGA}_{\Sigma\ CCS(c,X,Y)}$ and prnt_G is the parent map of G. This predicate describe an alternating sorting discipline *since it enforces the* *alternation between controls in A and the others which is required by the encoding given in [25, Def. 6.5]. This property is preserved by decomposition and by Proposition 4.7 φ is a weak monoidal predicate for $\odot$ on a presheaf describing bigraphs, therefore a bigraphical monoidal sorting.*

Example 4.9 (π-calculus)**.** *Milner proposed in [] an encoding of the π-calculus into binding (or local) bigraphs. An extension of pure bigraphs where names can be associated to localities. Binding bigraphs can be recovered from pure bigraphs by means of opportune sorting discipline that constrain the use of bounded resources in the link graph. This applies also to the encoding of the π. This encoding resembles the one given for the CCS except for the controls* send *and* get *which present an additional port, say the second one, to represent the actual and formal parameter of the communication i.e. the name sent. Therefore the signature to be used is $\Sigma_\pi = (\mathsf{alt}{:}0, \mathsf{send}{:}2, \mathsf{get}{:}2)$. Then, we just need to extends the alternate sorting discipline given for the CCS to check the use of bounded ports and names. This results in the definition of the predicate on the presheaf of bigraphs as the conjunction of the predicates:*

$$\varphi'_{(c,\langle m,X\rangle,\langle n,Y\rangle)}(G) = \bigwedge_{v\in\mathrm{dom}(c)} \mathrm{prnt}_G(v) \in \mathrm{dom}(c) \to (c(v) \in A \leftrightarrow \mathrm{prnt}_G(v) \notin A)$$

$$\varphi''_{(c,\langle m,X\rangle,\langle n,Y\rangle)}(G) = \bigwedge_{v\in\mathrm{dom}(c)} c(v) = \mathsf{recv} \to \mathrm{link}_G(2@v) \notin Y$$

$$\varphi'''_{(c,\langle m,X\rangle,\langle n,Y\rangle)}(G) = \bigwedge_{v\in\mathrm{dom}(c)} c(v) = \mathsf{recv} \to \forall u \in \mathrm{dom}(c).\forall i \in c(u).(\mathrm{link}_G(i@u) = 2@v \to v \in \mathrm{prnt}^*_G(u) \setminus \{u\})$$

where $A = \{\mathsf{send}, \mathsf{get}\}$, $(c, \langle m, X\rangle, \langle n, Y\rangle) \in \mathcal{B}_{\Sigma_\pi}$, $G \in \mathrm{B{\small GA}}_{\Sigma_\pi(c,\langle m,X\rangle,\langle n,Y\rangle)}$, link_G is the link map of G and $i@u$ denotes the i-th port of the node u. The first is the alternate discipline ensuring alternation between alt *and other controls along the parent map; the second ensures that second port of a node decorated with a* recv *control (i.e. the parameter of a receive) is not connected to an outer name which is indeed global; likewise, the last predicate ensures that every node connected to the second port of a node whose control is* recv *is a descendant of such a node and thus the use of the parameter is spatially bounded to the subtree rooted in the node declaring it. All these are bigraphical monoidal sortings like their conjunction.*

5 Reactive systems

Relative and idem-pushouts play a central role in the theory of reactive systems [25, Ch. 4]. In this section we show how RPOs and IPOs can be captured by presheaves characterized by particular constructions inside hom categories of automorphisms of M-Prof like $[\mathcal{B}_K, \mathrm{Set}]$ – presheaves can be thought as "generators" and constructions as "constraints" or "queries".

For conciseness of the presentation focus on monoids of these categories, but the proposed results also apply to partial monoids (as long as partiality is confined to the monoidal multiplication) with just few technicalities to be addressed to represent partial multiplications. However, the strategy used to model supported precategories (*cf.* Proposition 2.9) comes in help requiring only some footwork to be done. Moreover, since the following results rely only on properties of monoids and monoidal categories (and in some cases of topoi) they can be readily applied in other settings as long the target categories have enough structure.

In the following let $(\mathrm{A}, |-|)$ be a M-supported category described by a monoid (M, μ, η) of the hom category of M-Prof(C) and let $\odot$ and I denote tensor and unit of its monoidal structure.

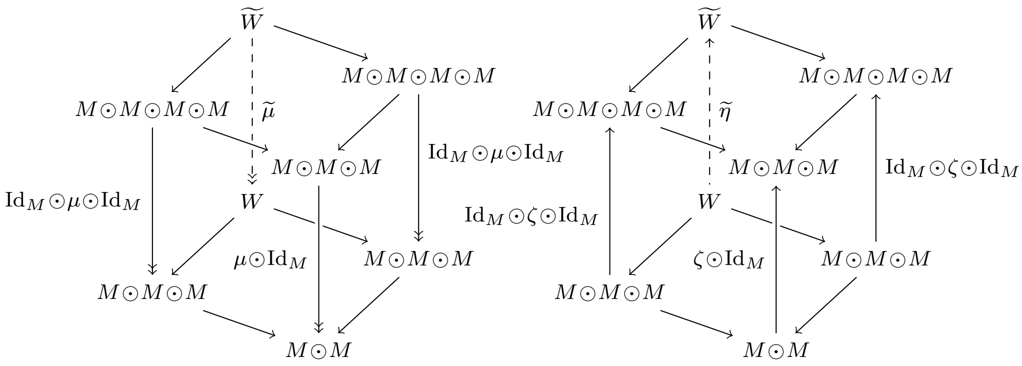
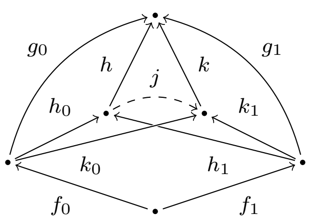


Figure 1: Relating $\widetilde{W}$ and W.

5.1 RPOs

Definition 5.1 (Relative pushout). *Let $\vec{f} = (f_0, f_1)$ and $\vec{g} = (g_0, g_1)$ be a span and a cospan forming a commuting square $g_0 \circ f_0 = g_1 \circ f_1$. A* bound for $\vec{f}$ relative to $\vec{g}$ *is a triple $(h, \vec{h})$ such that the cospan $\vec{h}$ form a commuting square with $\vec{f}$ and such that $g_0 = h \circ h_0$ and $g_1 = h \circ h_1$. A* relative pushout for $\vec{f}$ relative to $\vec{g}$ *is a bound $(h, \vec{h})$ (for $\vec{f}$ and relative to $\vec{g}$) such that for any other bound $(k, \vec{k})$ there exists a unique j such that $k \circ j = h$, $j \circ h_0 = k_0$ and $j \circ h_1 = k_1$ as shown by the diagram below.*



A category is said to have RPOs iff there exists an RPO for any commuting square.

Let W be the domain of the kernel pair of $(\mathrm{Id}_M \odot \mu) : (M \odot M \odot M) \to (M \odot M)$ (which exists because of presheaf categories being regular). W describes particular pairs of triples of elements generated by M modulo the equivalence relation induced by $\odot$. Elements of these pairs must produce the same image through $(\mathrm{Id}_M \odot \mu)$ and therefore

$$W_B \cong \{[(h, h', h'', f, g)] \mid [h \circ h' \circ f] = [h \circ h'' \circ g] \in M_B\}$$

for any stage B of the index category. In fact, the kernel pair of μ captures all commuting squares in the category A modeled by the monoid M and Id_M ensures that the first component of the triples is shared within each pair. Therefore, W describes every *RPO candidate* in A.

Let $\widetilde{W}$ denote the domain of the kernel pair of $(\mathrm{Id}_M \odot \mathrm{Id}_M \odot \mu)$. Intuitively, $\widetilde{W}$ composes another arrow of A to the candidates described by W. Because every arrow of a category at least factors through identities the monoidal multiplication of the monoid descibing it is an epimorphism. Then the epimorphism $(\mathrm{Id}_M \odot \mu \odot \mathrm{Id}_M)$ relates W and $\widetilde{W}$ and determines the unique map $\widetilde{\mu} : \widetilde{W} \twoheadrightarrow W$ as shown by the cube diagram in Figure 1 on the left. Because of the particular maps connecting the bases of the cube, structures in A described by $\widetilde{W}$ are particular decompositions of RPO candidates and every candidate described by W is the $\widetilde{\mu}$-image of every decomposition of the bound respect of which is defined. Specifically, $\widetilde{\mu}$ maps in $[(h, h', h'', f, g)]$ every $[(h, k, k', k'', f, g)]$ such that $h' = k \circ k'$ and $h'' = k \circ k''$ and intuitively tries to decompose the bound (h', h'') looking for any information shared between h' and h''. If, and only if, there exists only one of these decompositions (up-to iso) then the candidate it belongs is an RPO and k has to be an element of ηI_B for some stage B (*e.g.* an identity in A).

As illustrated by the diagram cube on the right of Figure 1, the pullbacks defining W and $\widetilde{W}$ are related also in the other way thanks to $\zeta : M \to M \odot M$ defined by the unitor $M \cong M \odot I$ and the unit $\eta : I \to M$. Therefore, there exists a unique morphism $\widetilde{\eta} : W \to \widetilde{W}$ mapping every RPO candidate of the like of $[(h, h', h'', f, g)]$ i nto its "free decomposition" $[(h, k, k', k'', f, g)]$ where k is an element of ηI_B for a suitable B.

This construction can be applied to any subobject V of W – with few minor changes to accommodate some technical details – to define $\widetilde{V}$, $\widetilde{\mu}_V$ and $\widetilde{\eta}_V$. Moreover, for any stage B, V_B and $\widetilde{V}_B$ are isomorphic to suitable subsets of W_B and $\widetilde{W}_B$ respectively and thus they describe selections of candidates and decompositions. These selections would contain only RPOs if, and only if, the only way to decompose their elements (through the above construction) is by means of elements described by the unit I at the proper stages.

Proposition 5.2 (Describing RPOs)**.** *A subpresheaf V of W describes only RPOs iff $\widetilde{\mu}_V$ and $\tilde{\eta}_V$ are inverses of each other i.e. $\widetilde{V} \cong V$.*

Proof. Consider a stage B and let $[(h, h', h'', f, g)] \in V_B$ be a class of RPO candidates but not actual RPOs. Suppose that $\widetilde{\mu}_V$ and $\widetilde{\eta}_V$ are inverses of each other. Because of (h, h', h'', f, g) not being an RPO there exists in $\widetilde{V}_B$ a decomposition $[(h, k, k', k'', f, g)]$ such that for every B' $k \notin I_{B'}$. It follows that (h, h', h'', f, g) is the $\widetilde{\mu}_V$-image of (h, k, k', k'', f, g) but (h, k, k', k'', f, g) is not the $\widetilde{\eta}_V$-image of (h, h', h'', f, g) and then $\mathrm{Id}_V \neq \widetilde{\mu}_V \widetilde{\eta}_V$ contradicting the assumption that $\tilde{\mu}_V$ and $\tilde{\eta}_V$ were inverses. Thus we are forced to conclude that V describes only RPOs.

For the converse, suppose that V describes only RPOs *i.e.* for any stage B, every $[(h, h', h'', f, g)] \in V_B$ contains only RPO. Hence the pre-images of (h, h', h'', f, g) through $\widetilde{\mu}_V$ are tuples of the like of (h, k, k', k'', f, g) where $k \in I_{B'}$ for some B'. Therefore $\widetilde{\mu}_V$ and $\widetilde{\eta}_V$ are inverses of each other. $\square$

Proposition 5.3 (Existence of RPOs)**.** *The supported category* A *described by* (M, μ, η) *has RPOs if, and only if, there exists a presheaf RPO satisfying Proposition 5.2 that makes diagram below commute.*

$$
\begin{tikzcd}
 & & M \odot M & & \\
 & M \odot M \odot M \arrow[ur, "\mathrm{Id}_M \odot \mu"] \arrow[dl, "\mu \odot \mathrm{Id}_M"'] & & M \odot M \odot M \arrow[ul, "\mathrm{Id}_M \odot \mu"'] \arrow[dr, "\mu \odot \mathrm{Id}_M"] & \\
 & & RPO \arrow[ul] \arrow[ur] & & \\
M \odot M & & & & M \odot M \\
 & & W \arrow[uu, two heads, "\xi"] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & M \odot M \odot M \arrow[uul, "\mu \odot \mathrm{Id}_M"] \arrow[dr, "\mathrm{Id}_M \odot \mu"'] & & M \odot M \odot M \arrow[uur, "\mu \odot \mathrm{Id}_M"'] \arrow[dl, "\mathrm{Id}_M \odot \mu"] & \\
 & & M \odot M & &
\end{tikzcd}
$$

Proof. Suppose A having RPOs, then for any commuting square in A there exists an RPO. The preasheaf *RPO* describes only RPOs because it satisfies Proposition 5.2 by hypothesis. In order to make the diagram above commute, $\xi : W \twoheadrightarrow RPO$ has to map every class of RPO candidates in the class of the RPO for the same span and bound *i.e.* commuting square. By construction, this class contains only RPOs. Reworded, ξ maps every candidate (h, h', h'', f, g) described by W in a RPO for the span (f, g) with bound $(h \circ h', h \circ h'')$. If *RPO* is not the presheaf of all RPOs, then it either does not satisfy Proposition 5.2 or there exists an RPO and a stage B such that it is not an element of any class in the set RPO_B. In every case the assumptions are contradicted and therefore if A has RPOs then the presheaf *RPO* above does exist.

For the converse, assume that the presheaf *RPO* exists and makes the above diagram commute. Clearly, every RPO is ξ-image of an RPO candidate for the same span and bound. Suppose that A does not have RPOs. It follows that there exists a span and a bound for it forming a commuting square which does not admit any RPO. Such a candidate would be described by W but cannot have a ξ-image such that the diagram commutes. Therefore A is forced to have RPOs. $\square$

If *RPO* exists, then every presheaf satisfying Proposition 5.2 is a subobject of *RPO* and all together constitute a lattice with the constantly empty presheaf as the bottom.

RPO could have been defined directly merging Propositions 5.2 and 5.3. However, the resulting diagram would have been quite complex – a sort of hypercube.

5.2 IPOs

Idem pushout can be thought as particular RPO and are often used to define "minimal" contexts for triggering reaction of a reactive system [25, Ch. 7] which are subject of Section 5.4.

Definition 5.4 (Idem pushout)**.** *Given a span $\vec{f}$, a cospan $\vec{g}$ is said to be an IPO for $\vec{f}$ iff* $(\mathrm{Id}_G, \vec{g})$ *is an RPO for $\vec{f}$ relative to $\vec{g}$ where G is the codomain of the cospan $\vec{g}$.*

IPO candidates can be seen as particular RPO candidates like IPOs are peculiar RPOs. It follows that a presheaf describing IPOs is a subpresheaf of the one describing RPOs and therefore it can be defined by means of the internal logic of the topos as a suitable subobject *RPO*. Given $[(h, h', h'', f, g)] \in RPO_B$ it would suffice to check whatever there exists a suitable B' such that $h \in I_{B'}$. However, the presheaf describing IPOs can be determined without resorting to this logic mimicking the approach of the previous section. To this end, let Q denote the domain of the kernel pair of $(\nu \odot \mu) : (I \odot M \odot M) \to (M \odot M)$. Q is a subobject of W and for any given stage B

$$Q_B \cong \{[(h, h', h'', f, g)] \mid [h \circ h' \circ f] = [h \circ h'' \circ g] \in M_B \wedge \exists B' \ h \in I_{B'}\}.$$

Therefore Q describes every RPO candidate (h, h', h'', f, g) (up-to the equivalence relation induced by $\odot$) such that the shared component h of the bound (h, h', h'') is an element of the unit I. Such tuples can be thought as *idem pushout candidates.* For instance, consider the monoid *Big* of $[\mathcal{B}_K, \mathrm{SET}]$, then $I \triangleq \mathrm{Id}_{\mathbb{N}\times\mathbb{B}}$ describes bigraphical terms composed by identities as placings and bijective renaming as wirings.

IPO, the presheaf of IPOs, can be defined from Q following the construction proposed to recover RPOs. However, it is easier to take advantage of the information already present in *RPO* and define *IPO* as the pullback of the monos $\iota_Q : Q \rightarrowtail W$ and $\iota_{RPO} : RPO \rightarrowtail W$ that characterize Q and *RPO* as subobjects of W.

$$\begin{tikzcd}
 & IPO \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" very near start] & \\
Q \arrow[dr, "\iota_Q"'] & & RPO \arrow[dl, "\iota_{RPO}"] \\
 & W &
\end{tikzcd}$$

The presheaves *IPO* and *RPO* describe every IPO and every RPO in a given stage covering every span of arrows in A. However they can be restricted to chosen classes of morphisms (up-to given arrows in the index category *e.g.* support translations). To this end let $\iota_F : F \rightarrowtail M$ and $\iota_G : G \rightarrowtail M$ be two subobjects of M. The pullback of $\mathrm{Id}_M \odot (\mu \circ (\mathrm{Id}_M \odot \iota_F))$ along $\mathrm{Id}_M \odot (\mu \circ (\mathrm{Id}_M \odot \iota_G))$ defines a subobject $W_{F,G}$ of W which, for every stage, is isomorphic to the set of RPO candidates for a span generated by the presheaf $\langle F, G\rangle$ – W can be seen as a shorthand for $W_{M,M}$.

$$\begin{tikzcd}
& W_{F,G} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
M\odot M\odot F \arrow[dr] & & M\odot M\odot G \arrow[dl] \\
& M\odot M &
\end{tikzcd}$$

RPOs and IPOs for these spans are described by $RPO_{F,G}$ and $IPO_{F,G}$ which are defined by pullback of $\iota_{W_{F,G}} : W_{F,G} \rightarrowtail W$ along ι_{RPO} and by pullback of ι_Q along $\iota_{RPO_{F,G}} : RPO_{F,G} \rightarrowtail W$ respectively:

$$\begin{tikzcd}
& RPO_{F,G} \arrow[dl] \arrow[dr] & \\
W_{F,G} \arrow[dr, tail, "\iota_{W_{F,G}}"'] & & RPO \arrow[dl, tail, "\iota_{RPO}"] \\
& W &
\end{tikzcd}
\qquad
\begin{tikzcd}
& IPO_{F,G} \arrow[dl] \arrow[dr] & \\
Q \arrow[dr, tail, "\iota_Q"'] & & RPO_{F,G} \arrow[dl, tail, "\iota_{RPO_{F,G}}"] \\
& W &
\end{tikzcd}$$

Example 5.5. *Let R be a subpresheaf of M. The presheaf $IPO_{M,R}$ describes all IPOs for any span (g, r) in* A *such that $r \in R_C$ for a suitable stage C. In particular, if R describes redexes of reaction rules (e.g. bigraphical reaction rules [25, Def. 8.5]) then $IPO_{M,R}$ characterizes all "minimal" (i.e. part of an IPO) contexts for these reaction rules. A presheaf describing such contexts can be defined by a suitable projection, say π_c. However, set of contexts returned by $\pi_c \circ IPO_{M,R}$ are in the "wrong" stage. This happens because, by definition of the presheaf of IPOs, the stage refers the whole square forming the IPO. This can be addressed defining the presheaf of minimal contexts for the redexes in R as*

$$M \sqcap \int^B (\pi_c \circ IPO_{M,R})(B)$$

where $\sqcap$ is the extension of set intersection to presheaf categories.

Relative pullbacks (RPB) and idempotent pullbacks (IPB) can be characterized with minor changes to the constructions presented so far. These changes roughly amount to swapping every $\odot$ in order to capture the symmetrical compositions. As an example, the presheaf of RPB candidates is defined as the domain of the kernel pair of $(\mu \odot \mathrm{Id}_M) : (M \odot M \odot m) \to (M \odot M)$. For any stage B, this presheaf is isomorphic to the set

$$\{[(f, g, h', h'', h)] \mid [f \circ h' \circ h] = [g \circ h'' \circ h] \in M_B\}$$

which is the set of all (classes of) RPB candidates whose compositions are in M_B. Likewise, the presheaf of RPB is defined mimicking the constructions illustrated in Figure 1 and Proposition 5.3

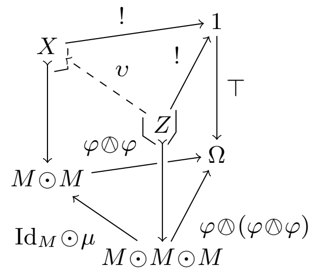
5.3 Weak monoidal predicates and RPOs

An important property of predicate sortings [10] is that they *transfer* RPOs. This means that for any given commuting square, the RPO determined by restricting to well-sorted morphisms (*e.g.* bigraphs) have to coincide with the one obtained considering unsorted ones. This result still holds in our setting for weak (partial) monoidal predicates and (partial) monoidal RPOs end is equivalent to state that the presheaf of RPOs defined by restricting to well-sorted objects is a subobject of that coming from unsorted ones.

Furthermore, the characteristic map of these presheaves are determined by the weak monoidal predicate considered.

Restating directly all the construction presented for RPOs and IPOs over the subpresheaf of M characterized by a given weak monoidal predicate will fail short to define well-sorted RPOs and IPOs. This is because such a subpresheaf only describes well-sorted arrows of A and do not offer any information about their compositions. However, both informations are contained in the predicate considered.

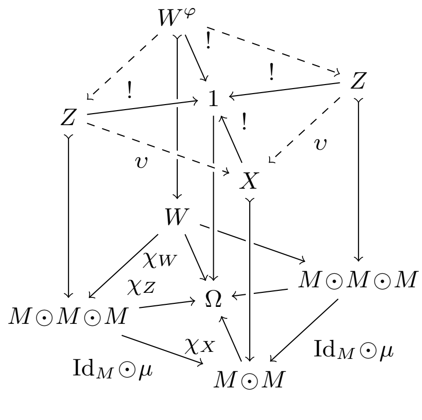
Let $\varphi : M \to \Omega$ be a weak monoidal predicate for M and let $\iota_Y : Y \rightarrowtail M$ and $\iota_X : X \rightarrowtail (M \odot M)$ denote the subobjects determined by it. Then there exists a unique $\nu : X \to Y$ by Definition 4.6. This morphism correspond to the restriction of the monoidal multiplication μ to φ-well-sorted elements generated by M *i.e.* arrows of A. Iterating the definition one obtain triples (and so on) of well-sorted arrows whose composition is again well-sorted thanks to the fact that any WMP is also a morphism between monoidal multiplications (here μ and $\oslash$). This defines $\iota_Z : Z \rightarrowtail (M \odot M \odot M)$ and $\upsilon : Z \to X$ such that the prism of pullbacks in the diagram below commutes.



The choice of $(\mathrm{Id}_M \odot \mu)$ is related to the definition of well-sorted RPOs and can be thought as the further decomposition of the "right side" of Y. In fact, υ maps every class of triples like (h, g, f) in the class of the pair $(h, g \circ f)$ ensuring that components and compositions are φ-well-sorted.

Let W^φ denote the domain of the kernel pair of υ. At every stage B, W^φ_B is isomorphic to the set of all the (class of) RPO candidates such that their components and their compositions are φ-well-sorted (Henceforth objects superscripted with a weak monoidal predicate will denote the induced subobject). By a further well-sorted decomposition of Z we define $\widetilde{W}^\varphi$ as the domain of the kernel pair of the restriction of composition to well-sorted quadruples of arrows. Then, the presheaves of φ-well-sorted RPOs, RPO^φ, is readily defined from W^φ and $\widetilde{W}^\varphi$. Likewise, we define IPO^φ, the presheaf of well-sorted IPOs.

All the construction involved can be seen as "projections" of the ones proposed for the unsorted case. Moreover these are all induced by the weak monoidal predicate φ as suggested by the commutative diagram below where every square is pullback.



This diagram defines W^φ as the subobject of W having characteristic map $\chi_{W^\varphi} = (\chi_Z \wedge \chi_Z) \circ \langle p_0, p_1 \rangle$ where χ_Z is the characteristic map of Z and $p_0, p_1 : W \to M \odot M \odot M$ are the kernel pair of $(\mathrm{Id}_M \odot \mu)$. $\chi_Z = \varphi \oslash \chi_X$ depends only on φ and χ_X which, by Definition 4.6 is equal to $\varphi \oslash \varphi$ and $\varphi \circ \mu$. Therefore, the previous construction is completely determined by the given weak monoidal predicate φ.

Weak monoidal predicates transfers monoidal RPOs if, and only, if, the presheaf of RPOs defined by restricting to well-sorted elements is a subobject of that coming from unsorted ones.

Proposition 5.6 (Transferring RPOs)**.** *Let φ be a weak monoidal predicate, then the presheaf RPO^φ of φ-well-sorted RPOs is a subobject of RPO, as IPO^φ is for IPO.*

Proof. By construction of W^φ (the presheaf of φ-well-sorted RPO candidates) the pullback of ι_{W^φ} along ι_{RPO} determines (up to isomorphism) the presheaf of φ-well-sorted RPOs where the monomorphism ι_{W^φ} is defined by the predicate $\chi_{W^\varphi} : W \to \Omega$. The presheaf IPO^φ is characterized in the same way by pulling back ι_{W^φ} along ι_{IPO}. Hence RPO^φ and IPO^φ are subpresheaves of RPO and IPO respectively. $\square$

5.4 Transitions

Dynamics for reactive systems are usually expressed by means of reaction rules. In particular, a parametric reaction rule for bigraphs [25, Def. 8.5] is a triple (r, r', ρ) where both the redex r and the reactum r' are bigraphs and the instantiation rule $\varrho \in \mathbb{F}^{op}$ ($\varrho \in \mathbb{I}^{op}$ for non duplicating rules) is a map from the sites of the redex to the reactum ones as illustrated by the following diagram.

$$\begin{tikzcd}
 & I_1 & \\
I_0 \arrow[ur, "r"] \arrow[rr, dotted, bend right, "\varrho"'] & & I_2 \arrow[ul, "r'"']
\end{tikzcd}$$

The instantiation rule describes how the parameters of the redex are instantiated in the reactum. Then there is a transition $a \rightarrowtriangle a'$ for the given reaction rule if and only if both a and a' are agents (*i.e.* bigraph whose inner interface is empty) $a = c \circ r \circ d$ and $a' = c \circ r' \circ \varrho(d)$ where $\varrho(d)$ "selects and instantiates" the parameters of the rule. The bigraphs c and d are said *context* and *parameter* of the rule respectively.

This format of reaction or parametric rewriting is not limited to bigraphs but is at least available to any reactive system modeled by a supported (pre)category. However, it is a special case of the following format where the map between the outer interfaces is forced to be an identity (thus characterizing rewritings as cospans).

$$\begin{tikzcd}
I_2 & & I_3 \arrow[ll, dotted, bend right] \\
I_0 \arrow[u, "r"] \arrow[rr, dotted, bend right] & & I_1 \arrow[u, "r'"']
\end{tikzcd}$$

Without loss of generality let redexes and reacti be arrows of an M-supported category $(\mathrm{A}, |-|)$ – the case of supported precategories is an extension to partial monoids of the following results – and let the monoid (M, μ, η) of the hom-category of M-Prof(C) describe A. Then, redexes, reacti and instantiation maps are characterized by suitable subpresheaves R and R' of M and an endofunctor ρ over the index category. Taking ρ to be a generic endofunctor introduce a further relaxation on the instantiation maps, but renders the actions on supports explicit adding useful information *e.g.* which nodes of a bigraph are preserved, deleted or created (in contrast with the above formulations). A similar approach can be found in [8, 13, 17] This additional data is indeed valuable from an implementation perspective since it allows to trace information during reaction *e.g.* to not recompute unaltered regions of a bigraph. In any case, it is always possible to restrict to particular classes of endofunctor, for instance to capture non duplicating rules.

Definition 5.7 (Permissive reaction rule)**.** *Given an* M*-supported category* $(\mathrm{A}, |-|)$ *modeled by the monoid* (M, μ, η) *of the hom-category of* M-PROF(C)*, a* permissive reaction rule θ *is a triple* (R, R', ρ) *where* R *and* R' *are non empty subpresheaves of* M *and* ρ *is an endofunctor over the index category of these presheaves. The rule is said to be* meaningful *iff for every stage* B $R_B = \emptyset \Leftrightarrow R'_{\rho B} = \emptyset$.

A rewriting is roughly the "substitution" of $r \in R_B$ with $r' \in R'_{B'}$ for $B' = \rho_B$ with ρ acting as a glue. This is described by the pullback of the terminal map from R along the terminal one from $(R' \circ \rho)$ which, at every stage B, is isomorphic to the set of pair (r, r'). These substitutions extend to every arrow $a = c \circ r \circ d$ in the category A (which are described by the presheaf $M \odot R \odot M$) and $a' = c' \circ r' \circ d'$ $(M \odot (R' \circ \rho) \odot M)$ accordingly to ρ which, because of being defined on supports and objects of A (*i.e.* interfaces), contains enough information to define a "way to put" the reactum into the place of the redex and a "way to instantiate" parameters and contexts to fit the reactum. (How this translates to the system modeled by A obviously depends on the contingent modelization.)

$$\begin{tikzcd}
I_5 \arrow[rrr, dash, dotted, bend left] & & & I_5 \\
& I_2 \arrow[ul, "c"'] & I_3 \arrow[l, dotted] \arrow[ur, "c'"] & \\
& {} \arrow[r, Rightarrow, "\varrho"] & {} & \\
& I_0 \arrow[uu, "r"'] \arrow[r, dotted] & I_1 \arrow[uu, "r'"] & \\
I_4 \arrow[uuuu, "a"] \arrow[ur, "d"'] \arrow[rrr, dash, dotted, bend right] & & & I_4 \arrow[uuuu, "a'"'] \arrow[ul, "d'"]
\end{tikzcd}$$

This is captured by the following pullback.

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
M \odot R \odot M \arrow[dr, "\mathrm{Id}_M \odot ! \odot \mathrm{Id}_M"'] & & M \odot (R' \circ \rho) \odot M \arrow[dl, "\mathrm{Id}_M \odot ! \odot \mathrm{Id}_M"] \\
& M \odot 1 \odot M &
\end{tikzcd}$$

For any stage B, the presheaf defined by this pullback returns a set isomorphic to a set whose elements describe (up-to the equivalence relation induced by the coend defining $\odot$) context and parameter for each admitted pair of redex and reactum:

$$Z_B \cong \{[(c, d, r, r')] \mid c \circ r \circ d \in M_B r \in R_C \wedge r' \in R'_{\rho C}\}.$$

Redex embeddings – *i.e.* factorization of the arrows of A into context and parameter – are defined likewise by the presheaf obtained by pulling back Id_M along $\mu \circ (\mathrm{Id}_M \odot \mu)$ or (by associativity of the monoidal multiplication) $\mu \circ (\mu \odot \mathrm{Id}_M)$.

$$\begin{tikzcd}
& Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
M \arrow[dr, "\mathrm{Id}_M"'] & & M \odot R \odot M \arrow[dl, "\mu \circ (\mathrm{Id}_M \odot \mu)"] \\
& M &
\end{tikzcd}$$

Then, for any stage B, $Y_B \cong \{(a,c,d,r) \mid a \in M_B \wedge a = c \circ r \circ d\}$. In order to describe reaction for any arrow a of A it suffices to compose the pullbacks above on $M \odot R \odot M$ as illustrated by the following diagram:

$$\begin{tikzcd}[column sep=small]
 & & W \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
M \arrow[dr] & & M \odot R \odot M \arrow[dl] \arrow[dr] & & M \odot (R' \circ \rho) \odot M \arrow[dl] \\
 & M & & M \odot 1 \odot M &
\end{tikzcd}$$

$$W_B \cong \{(a,c,d,r,r') \mid a \in M_B \wedge a = c \circ r \circ d \wedge r \in R_C \wedge r' \in R'_{\rho C}\}.$$

In some scenarios may be needed to allow reaction only for specific arrows of A; for instance bigraphical reactive systems are defined only on agents – *i.e.* bigraphs having the empty interface as their inner interface. This can be readily captured with the proposed approach: let N be the subpresheaf of M defining the desired restriction then just replace N for every M representing parameters or the arrows to be factored in the definitions above. This leads to the definition of subpresheaves of W and hence to sub-reactive systems of the one naturally induced by the rules (*i.e.* the one described by W). Clearly this still holds for weak monoidal predicates and in particular for predicate sortings.

The same approach leads to the definition of reaction whose contexts are minimal in the sense of the idem pushout construction (*cf.* Example 5.5).

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
IPO_{M,R} \arrow[dr, swap, "(\mathrm{Id}_M \odot !) \circ \pi_{c,d}"] & & M \odot (R' \circ \rho) \arrow[dl, "\mathrm{Id}_M \odot !"] \\
 & M \odot 1 &
\end{tikzcd}$$

For any stage B, Q_B is isomorphic to the set of classes of tuples (b,c,e,r,r') such that (c,e) is an IPO for the span (b,r) in A and that r' is a reactum for r in θ.

The definitions above have been stated for a single reaction rule, but readily extends to sets of reaction rules: it suffices to join all the presheaves obtained for each rule separately.

5.5 Bigraphical reactive systems

To recast the theory of bigraphical reactive systems (BRS) into the proposed setting it suffices to apply the results presented so far to a (partial) monoid describing bigraphs. Without any loss of generality we focus on bigraphs with abstract names (Section 3.4).

Bigraphs with abstract names over a given signature K forms a $\mathcal{C}_K$-supported category which is modeled by the monoidal presheaf $\mathrm{B{\scriptsize GA}}_K$. Then, to recover BRSs and IPOs for these bigraphs we only have to replace $\mathrm{B{\scriptsize GA}}_K$ for the category A and the monoid M in the above definitions; and restrict the endofunctors used in Definition 5.7 to match bigraphical reaction rules [25, Def. 8.5].

An instantiation map for a bigraphical reaction rule is a map between sites but redex and reactum implicitly defines also a partial function between control maps. Let $\mathcal{C}'_K \triangleq I_{\mathbb{F}+\mathbb{F}} \downarrow U_{\mathcal{K}} K$ be the category having the same objects of $\mathcal{C}_K$ but partial maps as morphisms and let $I^{\mathcal{F}}_{\mathcal{B}_K} : \mathcal{B}_K \rightarrowtail \mathcal{F}$ be the injection functor from $\mathcal{B}_K$ to $\mathcal{F} \triangleq \mathcal{C}'_K \times (\mathbb{F} \times \mathbb{B})^{op} \times (\mathbb{N} \times \mathbb{B})$. Then an instantiation map is an object of the comma category $I^{\mathcal{F}}_{\mathcal{B}_K} \downarrow I^{\mathcal{F}}_{\mathcal{B}_K}$. In particular, a non-duplicating instantiation rule is an object of $I^{\mathcal{I}}_{\mathcal{B}_K} \downarrow I^{\mathcal{I}}_{\mathcal{B}_K}$ where $I^{\mathcal{I}}_{\mathcal{B}_K} : \mathcal{B}_K \rightarrowtail \mathcal{I}$ is the injection functor from $\mathcal{B}_K$ to $\mathcal{I} \triangleq \mathcal{C}'_K \times (\mathbb{I} \times \mathbb{B})^{op} \times (\mathbb{N} \times \mathbb{B})$. It is easy to check that every instantiation map defines an endofunctor over $\mathcal{B}_K$ mapping the type of the redex into the reactum's one and leaving everything else untouched. In particular, it is an object of the comma $I^{\mathcal{F}}_{\mathcal{B}_K} \downarrow I^{\mathcal{F}}_{\mathcal{B}_K}$.

Definition 5.8. *A* bigraphical reaction rule θ *is a permissive reaction rule for* $\textsc{Bga}_K$ *such that the endofunctor is an object of* $I^{\mathcal{F}}_{\mathcal{B}_K} \downarrow I^{\mathcal{F}}_{\mathcal{B}_K}$ *The rule is called* non duplicating *iff* $\rho \in I^{\mathcal{I}}_{\mathcal{B}_K} \downarrow I^{\mathcal{I}}_{\mathcal{B}_K}$ *and is said to be* ground *iff* $R_{(C,X,Y)} = \emptyset \Leftrightarrow X \neq \varepsilon$ *otherwise it is said to be* parametric.

Let $\theta = (R, R', \rho)$ be a bigraphical reaction rule. Then, reactions with minimal (in the sense of IPOs) contexts are defined by the pullback below.

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
IPO_{\textsc{Bga}_K,R} \arrow[dr, "(\mathrm{Id}_{\textsc{Bga}_K} \odot !) \circ \pi_{c,d}"'] & & \textsc{Bga}_K \odot (R' \circ \rho) \arrow[dl, "\mathrm{Id}_{\textsc{Bga}_K} \odot !"] \\
 & \textsc{Bga}_K \odot 1 &
\end{tikzcd}$$

Let $G \hookrightarrow \textsc{Bga}_K$ characterize ground bigraphs in $\textsc{Bga}_K$. Clearly $\textsc{Bga}_K \odot G$ and $G \odot \textsc{Bga}_K$ are (surjectively) mapped into G by the monoidal multiplication of $\textsc{Bga}_K$. The bigraphical reactive system (on ground bigraphs with abstract names) for the given rule θ is defined by the following diagram.

$$\begin{tikzcd}
 & & BRS_\theta \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
 & Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
G \arrow[dr] & & \textsc{Bga}_K \odot R \odot G \arrow[dl] \arrow[dr] & & \textsc{Bga}_K \odot (R' \circ \rho) \odot G \arrow[dl] \\
 & G & & \textsc{Bga}_K \odot 1 \odot G &
\end{tikzcd}$$

6 Conclusions and future works

In this paper we have given a new categorical presentation of precategories and s-categories as particular presheaves, i.e., profunctors. This allows to study these objects and their relations by leveraging the rich theory of presheaf categories. In particular, we have shown how graphical structures can be composed to yield (pluri-)graphical structures, like link and place graphs can be merged to yield bigraphs. Within this presheaf category, we have shown that a language for describing bigraphs arises naturally as a free algebra over basic elements (ions, places and wirings) and we have introduced a class of *weak monoidal predicates* for specifying sorting disciplines. Remarkably, these predicates are definable in the internal language of the same presheaf category (which is a topos), pointing out that this can be the right context for the interpretation of logics for decomposable predicates, like the one introduced in [26]. Our categorical presentation allows also for the characterization of RPOs, IPOs and other results about rewriting systems, thus covering also the dynamic part of bigraphs. All of these within the same category.

A future work worth investigating is the representation of *vertical refinements* in presheaf categories. In fact, taking $\mathcal{K}$ to be the free category with finite coproducts over $\mathbb{N}$ makes cod : $\mathcal{C} \to \mathcal{K}$ (where $\mathcal{C} = I_{\mathbb{B}} \downarrow U_{\mathcal{K}}$) an opfibration. This offers a formal way to propagate operation over signatures (*e.g.* unification of controls) to supports and eventually presheaves, thus paving the way for formalizing refinements for systems and languages modeled in these categories.

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A Preliminaries

A bicategory is a particular algebraic notion of weak 2-category. The idea is that a bicategory is a category weakly enriched over CAT: the hom-objects of a bicategory are hom-categories, but the associativity and unity laws of enriched categories hold only up to coherent isomorphism.

Definition A.1 (Bicategory). *A bicategory* B *consists of*

(1) *a collection of objects, also called* 0-cells*;*

(2) *for each pair of 0-cells X and Y, a category* $\mathrm{B}(X,Y)$*, whose objects are called morphisms or* 1-cells *and whose morphisms are called 2-morphisms or* 2-cells*;*

(3) *for each 0-cell X, a distinguished 1-cell* $\mathrm{Id}_X \in \mathrm{B}(X,X)$ *called the identity morphism or identity 1-cell at X;*

(4) *for each triple of 0-cells X, Y and Z, a functor* $\circ : \mathrm{B}(Y,Z) \times \mathrm{B}(X,Y) \to \mathrm{B}(X,Z)$ *called* horizontal composition*;*

(5) *for each pair of 0-cells X and Y, natural isomorphisms called* unitors*:* $\mathrm{Id}_{\mathrm{B}(X,Y)} \circ \mathcal{K}_{\mathrm{Id}_X} \cong \mathrm{Id}_{\mathrm{B}(X,Y)} \cong \mathcal{K}_{\mathrm{Id}_Y} \circ \mathrm{Id}_{\mathrm{B}(X,Y)} : \mathrm{B}(X,Y) \to \mathrm{B}(X,Y)$*;*

(6) *for each quadruple of 0-cells W, X, Y and Z, a natural isomorphism called the* associator *between the two functors from* $\mathrm{B}(Y,Z) \times \mathrm{B}(X,Y) \times \mathrm{B}(W,X)$ *to* $\mathrm{B}(W,Z)$ *built out of $(- \circ -)$ such that the same axioms as the constraint isomorphisms in a monoidal category (which we do not write out in full here) are satisfied.*

Proposition A.2 (Day's convolution [9]). *Given a (symmetric) monoidal category* $(\mathrm{M}, \otimes, \epsilon)$*, the Yoneda embedding $\mathcal{Y}$ is a strong monoidal functor defining the (symmetric) closed monoidal category* $(\mathrm{SET}^{\mathrm{M}^{op}}, \star, \mathcal{Y}\epsilon)$

Proof. The tensor $\star$ is defined as the left Kan extension along $\mathcal{Y}\times\mathcal{Y}$ of $\mathcal{Y}(-\otimes-)$ illustrated by the diagram below.

$$\begin{tikzcd}
\mathrm{M} \times \mathrm{M} \arrow[r, "\mathcal{Y} \times \mathcal{Y}"] \arrow[d, "\mathcal{Y}(- \otimes -)"'] & \mathrm{SET}^{\mathrm{M}^{op}} \times \mathrm{SET}^{\mathrm{M}^{op}} \arrow[dl, "Lan_{\mathcal{Y}\times\mathcal{Y}}\mathcal{Y}(- \otimes -)"] \\
\mathrm{SET}^{\mathrm{M}^{op}} &
\end{tikzcd}$$

Then, the definition of $\star$ can be unfolded as:

$$X \star Y \triangleq \int^{M,N} (X \times Y)(M,N) \times \mathrm{M}(-, M \otimes N)$$

$$X \star Y \cong \left(\coprod_{M,N} (X \times Y)(M,N) \times \mathrm{M}(-, M \otimes N) \right)_{/\approx}$$

where $\approx$ is the relation defined as:

$$(x, y, q : Q \to M \otimes N) \approx (x', y', q' : Q \to M' \otimes N') \iff \exists m \in \mathrm{M}(M, M'), n \in \mathrm{M}(N, N') : \\ (X \times Y)(m \times n)(x', y') = (x, y) \wedge \mathrm{M}(Q, m \otimes n)(q) = q'.$$

By application of the co-Yoneda lemma on the two coends above $\mathcal{Y}\epsilon$ is the monoidal unit. Then, coherence conditions, closure and the natural isomorphism $\phi_{M,N} : \mathcal{Y}M \star \mathcal{Y}N \cong \mathcal{Y}(M \otimes N)$ follow are straightforward. ☐